

تمرين 1 : احسب النهايات التالية :

$$\lim_{x \rightarrow +\infty} \frac{|x-10| + x^2}{2x^2 + 5}$$

$$\lim_{x \rightarrow 1} \frac{|x-1| + x^2 - 1}{x^2 + x - 2}$$

$$\lim_{x \rightarrow 1} \left(\frac{\sqrt{x+3} + \sqrt{x} - 3}{x-1} \right)$$

$$\lim_{x \rightarrow 2} \frac{|x-1| + x^3 - 9}{x^2 - 4}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x \cos 2x}{x^2}$$

$$\lim_{x \rightarrow 1} \frac{\sin(f x)}{x-1}$$

$$\lim_{x \rightarrow \frac{f}{4}} \frac{x - \frac{f}{4} \tan^2 x}{x - \frac{f}{4}}$$

$$\lim_{x \rightarrow 1} \frac{x^3 \sqrt{x+3} - 2}{x-1}$$

تمرين 2 : احسب النهايات التالية :

$$\lim_{x \rightarrow 0} \frac{E(2x)}{x}$$

$$\lim_{x \rightarrow +\infty} \frac{E(2x)}{x}$$

$$\lim_{x \rightarrow 1} E(2x) + \sqrt{x}$$

$$\lim_{x \rightarrow 1} E(x) + x$$

$$\lim_{x \rightarrow 1} E(x^2) - (E(x))^2$$

$$\lim_{x \rightarrow +\infty} \frac{E(2x)}{E(x)}$$

تمرين 3 : احسب النهايات التالية :

$$\lim_{x \rightarrow +\infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{|\cos 3x - \cos x|}}{x}$$

$$\lim_{x \rightarrow \frac{f}{4}} \left(\frac{\sqrt{1 - \cos(x)} - \sqrt{1 - \sin(x)}}{1 - \tan(x)} \right)$$

$$\lim_{x \rightarrow -\infty} x\sqrt{x^2 + 1} + x^2$$

تمرين 4 :

1) احسب النهاية $\lim_{x \rightarrow a} \frac{x\sqrt{x} - a\sqrt{a}}{\sqrt{x} - \sqrt{a}}$ ($a \in [0; +\infty[$)

2) احسب النهاية $\lim_{x \rightarrow 1} \frac{x^{2015} - 1 - 2015(x-1)}{(x-1)^2}$

تمرين 1 :

$$\lim_{\substack{x \rightarrow 1 \\ x > 1}} \frac{|x-1| + x^2 - 1}{x^2 + x - 2} = \lim_{\substack{x \rightarrow 1 \\ x > 1}} \frac{x-1 + x^2 - 1}{x^2 + x - 2} = \lim_{\substack{x \rightarrow 1 \\ x > 1}} \frac{x^2 + x - 2}{x^2 + x - 2} = 1$$

$$\lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{|x-1| + x^2 - 1}{x^2 + x - 2} = \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{1-x + x^2 - 1}{x^2 + x - 2} = \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{x^2 - x}{x^2 + x - 2} = \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{x(x-1)}{(x-1)(x+2)} = \lim_{\substack{x \rightarrow 1 \\ x < 1}} \frac{x}{x+2} = \frac{1}{3}$$

$$\lim_{x \rightarrow +\infty} \frac{|x-10| + x^2}{2x^2 + 5} = \lim_{x \rightarrow +\infty} \frac{x-10 + x^2}{2x^2 + 5} = \lim_{x \rightarrow +\infty} \frac{x^2}{2x^2} = \frac{1}{2} \quad \text{لدينا : } \forall x \in [10; +\infty[\quad x-10 \geq 0$$

يمكننا دائما اعتبار $x \in [a; +\infty[$ عند حساب نهاية في $+\infty$
يمكننا دائما اعتبار $x \in]-\infty; a]$ عند حساب نهاية في $-\infty$

لدينا : $\forall x \in [1; 3] \quad x-1 \geq 0$

$$\lim_{x \rightarrow 2} \frac{|x-1| + x^3 - 9}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{x-1 + x^3 - 9}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{x^3 - 8 + x - 2}{x^2 - 4}$$

$$\lim_{x \rightarrow 2} \frac{|x-1| + x^3 - 9}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4 + 1)}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{x^2 + 2x + 5}{x+2} = \frac{13}{4}$$

يمكننا دائما اعتبار $x \in [a, b]$ حيث $x_0 \in [a, b]$ عند حساب نهاية في x_0

$$\lim_{x \rightarrow 1} \left(\frac{\sqrt{x+3} + \sqrt{x} - 3}{x-1} \right) = \lim_{x \rightarrow 1} \left(\frac{\sqrt{x+3} - 2}{x-1} + \frac{\sqrt{x} - 1}{x-1} \right) = \lim_{x \rightarrow 1} \left(\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} + \frac{x-1}{(x-1)(\sqrt{x}+1)} \right)$$

$$\lim_{x \rightarrow 1} \left(\frac{\sqrt{x+3} + \sqrt{x} - 3}{x-1} \right) = \lim_{x \rightarrow 1} \left(\frac{1}{\sqrt{x+3}+2} + \frac{1}{\sqrt{x}+1} \right) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$$\lim_{x \rightarrow 1} \frac{\sin(f(x))}{x-1} = \lim_{t \rightarrow 0} \frac{\sin(f(t+1))}{t} = \lim_{t \rightarrow 0} \frac{\sin(f(t+f))}{t} = \lim_{t \rightarrow 0} f \frac{\sin(f(t))}{f(t)} = f \quad (\text{نضع } t = x-1)$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x \cos 2x}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x + \cos x - \cos x \cos 2x}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} + \cos x \frac{1 - \cos 2x}{x^2}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x \cos 2x}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} + 4 \cos x \frac{1 - \cos 2x}{(2x)^2} = \frac{1}{2} + 4 \times 1 \times \frac{1}{2} = \frac{5}{2}$$

$$\lim_{x \rightarrow 1} \frac{x^3 \sqrt{x+3} - 2}{x-1} = \lim_{x \rightarrow 1} \frac{x^3(\sqrt{x+3} - 2 + 2) - 2}{x-1} = \lim_{x \rightarrow 1} \frac{x^3(\sqrt{x+3} - 2)}{x-1} + \frac{2x^3 - 2}{x-1}$$

$$\lim_{x \rightarrow 1} \frac{x^3 \sqrt{x+3} - 2}{x-1} = \lim_{x \rightarrow 1} x^3 \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} + 2 \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x \rightarrow 1} x^3 \frac{1}{\sqrt{x+3}+2} + 2(x^2+x+1)$$

$$\lim_{x \rightarrow 1} \frac{x^3 \sqrt{x+3} - 2}{x-1} = \frac{1}{4} + 6 = \frac{25}{4}$$

$$\lim_{x \rightarrow \frac{f}{4}} \frac{x - \frac{f}{4} \tan^2 x}{x - \frac{f}{4}} = \lim_{x \rightarrow \frac{f}{4}} \frac{x - \frac{f}{4} + \frac{f}{4} - \frac{f}{4} \tan^2 x}{x - \frac{f}{4}} = \lim_{x \rightarrow \frac{f}{4}} 1 + \frac{f}{4} \frac{1 - \tan^2 x}{x - \frac{f}{4}} = \lim_{x \rightarrow \frac{f}{4}} 1 - \frac{f}{4} (\tan x + 1) \frac{\tan x - 1}{x - \frac{f}{4}}$$

$$\lim_{x \rightarrow \frac{f}{4}} \frac{x - \frac{f}{4} \tan^2 x}{x - \frac{f}{4}} = \lim_{x \rightarrow \frac{f}{4}} 1 - \frac{f}{4} (1 + \tan x) \frac{\tan\left(x - \frac{f}{4}\right)(1 + \tan x)}{x - \frac{f}{4}} = \lim_{x \rightarrow \frac{f}{4}} 1 - \frac{f}{4} (1 + \tan x)^2 \frac{\tan\left(x - \frac{f}{4}\right)}{x - \frac{f}{4}}$$

$$\lim_{x \rightarrow \frac{f}{4}} \frac{x - \frac{f}{4} \tan^2 x}{x - \frac{f}{4}} = 1 - \frac{f}{4} \times 4 \times 1 = 1 - f$$

للتذكير: $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$ ما يعني أن $\tan(x - y) = \tan x - \tan y$

تمرين 2:

نعلم أن: $\forall x > 0$ $2x - 1 < E(2x) \leq 2x$ منه $\forall x > 0$ $2 - \frac{1}{x} < \frac{E(2x)}{x} \leq 2$ بما أن:

$$\lim_{x \rightarrow +\infty} 2 - \frac{1}{x} = \lim_{x \rightarrow +\infty} 2 = 2$$

$$\lim_{x \rightarrow +\infty} \frac{E(2x)}{x} = 2 \text{ فإن}$$

لدينا: $\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{E(2x)}{x} = \lim_{\substack{x \rightarrow 0 \\ x > 0}} 0 = 0$ منه: $x \in]0; \frac{1}{2}[\Rightarrow 2x \in]0; 1[\Rightarrow E(2x) = 0 \Rightarrow \frac{E(2x)}{x} = 0$

و لدينا: $\lim_{\substack{x \rightarrow 0 \\ x < 0}} \frac{E(2x)}{x} = \lim_{\substack{x \rightarrow 0 \\ x < 0}} \frac{-1}{x} = +\infty$ منه: $x \in]-\frac{1}{2}; 0[\Rightarrow 2x \in]-1; 0[\Rightarrow E(2x) = -1 \Rightarrow \frac{E(2x)}{x} = \frac{-1}{x}$

تمعن جيدا في الطريقة المتبعة في كل نهاية، فدالة الجزء الصحيح دالة خاصة جدا لذلك نستعمل طرقا خاصة بها.

عند حساب النهاية يمين عدد x_0 يمكنك دائما أن تعتبر $x \in]x_0; a[$ (أيضا على اليسار $x \in]a; x_0[$)

لدينا: $\lim_{\substack{x \rightarrow 1 \\ x > 1}} E(x) + x = \lim_{\substack{x \rightarrow 1 \\ x > 1}} 1 + x = 2$ منه: $x \in]1; 2[\Rightarrow E(x) = 1 \Rightarrow E(x) + x = 1 + x$

و $\lim_{\substack{x \rightarrow 1 \\ x < 1}} E(x) + x = \lim_{\substack{x \rightarrow 1 \\ x < 1}} x = 1$ منه: $x \in]0; 1[\Rightarrow E(x) = 0 \Rightarrow E(x) + x = x$

لا يمكننا التعويض ببساطة كما نفعل مع جل الدوال الاعتيادية، فدالة الجزء الصحيح نقول عنها في الرياضيات أنها دالة غير متصلة

لدينا: $x \in]1; \frac{3}{2}[\Rightarrow 2x \in]2; 3[\Rightarrow E(2x) = 2 \Rightarrow E(2x) + \sqrt{x} = 2 + x$

$$\lim_{\substack{x \rightarrow 1 \\ x > 1}} E(2x) + \sqrt{x} = \lim_{\substack{x \rightarrow 1 \\ x > 1}} 2 + x = 3 \text{ منه:}$$

و $\lim_{\substack{x \rightarrow 1 \\ x < 1}} E(2x) + \sqrt{x} = \lim_{\substack{x \rightarrow 1 \\ x < 1}} 1 + x = 2$ منه: $x \in]\frac{1}{2}; 1[\Rightarrow 2x \in]1; 2[\Rightarrow E(2x) = 1 \Rightarrow E(2x) + \sqrt{x} = 1 + x$

نعلم أن: $\forall x > 0$ $\begin{cases} 2x - 1 < E(2x) \leq 2x \\ \frac{1}{x} < \frac{1}{E(x)} \leq \frac{1}{x-1} \end{cases}$ منه: $\forall x > 0$ $\begin{cases} 2x - 1 < E(2x) \leq 2x \\ x - 1 < E(x) \leq x \end{cases}$

منه: $\forall x > 1$ $\frac{2x-1}{x} < \frac{E(2x)}{E(x)} \leq \frac{2x}{x-1}$ وبما أن: $\lim_{x \rightarrow +\infty} \frac{2x-1}{x} = \lim_{x \rightarrow +\infty} \frac{2x}{x-1} = 2$ فإن: $\lim_{x \rightarrow +\infty} \frac{E(2x)}{E(x)} = 2$

$$x \in]1; \sqrt{2}[\Rightarrow \begin{cases} x^2 \in]1; 2[\\ x \in]1; 2[\end{cases} \Rightarrow \begin{cases} E(x^2) = 1 \\ E(x) = 1 \end{cases} \Rightarrow E(x^2) - (E(x))^2 = 1 - 1^2 = 0 \quad \text{لدينا :}$$

$$\lim_{\substack{x \rightarrow 1 \\ x > 1}} E(x^2) - (E(x))^2 = \lim_{\substack{x \rightarrow 1 \\ x > 1}} 0 = 0 \quad \text{منه :}$$

$$x \in]0; 1[\Rightarrow \begin{cases} x^2 \in]0; 1[\\ x \in]0; 1[\end{cases} \Rightarrow \begin{cases} E(x^2) = 0 \\ E(x) = 0 \end{cases} \Rightarrow E(x^2) - (E(x))^2 = 0 - 0^2 = 0 \quad \text{لدينا :}$$

$$\lim_{x \rightarrow 1} E(x^2) - (E(x))^2 = 0 \quad \text{خلاصة :} \quad , \quad \lim_{\substack{x \rightarrow 1 \\ x < 1}} E(x^2) - (E(x))^2 = \lim_{\substack{x \rightarrow 1 \\ x < 1}} 0 = 0 \quad \text{منه :}$$

تمرين 3 : احسب النهايات التالية :

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{\sqrt{|\cos 3x - \cos x|}}{x} = \lim_{\substack{x \rightarrow 0 \\ x > 0}} \sqrt{\frac{|\cos 3x - \cos x|}{x^2}} = \lim_{\substack{x \rightarrow 0 \\ x > 0}} \sqrt{\frac{\cos 3x - 1}{x^2} + \frac{1 - \cos x}{x^2}}$$

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{\sqrt{|\cos 3x - \cos x|}}{x} = \lim_{\substack{x \rightarrow 0 \\ x > 0}} \sqrt{-9 \frac{1 - \cos 3x}{(3x)^2} + \frac{1 - \cos x}{x^2}} = \sqrt{\frac{-9}{2} + \frac{1}{2}} = 2$$

$$\lim_{\substack{x \rightarrow 0 \\ x < 0}} \frac{\sqrt{|\cos 3x - \cos x|}}{x} = \lim_{\substack{x \rightarrow 0 \\ x < 0}} -\sqrt{\frac{|\cos 3x - \cos x|}{x^2}} = -2$$

$$\forall x < 0 \quad x = -x^2 \quad \text{و} \quad \forall x > 0 \quad x = x^2$$

$$\lim_{x \rightarrow +\infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x + \sqrt{x}}}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x \left(1 + \frac{\sqrt{x}}{x}\right)}}{\sqrt{x \left(1 + \frac{\sqrt{x + \sqrt{x}}{x}\right)} + \sqrt{x}}$$

$$\lim_{x \rightarrow +\infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x} \sqrt{1 + \frac{\sqrt{x}}{x}}}{\sqrt{x} \left(\sqrt{1 + \frac{\sqrt{x + \sqrt{x}}{x^2}} + 1} \right)} = \lim_{x \rightarrow +\infty} \frac{\sqrt{1 + \frac{1}{\sqrt{x}}}}{\sqrt{1 + \frac{1}{x} + \frac{1}{x\sqrt{x}} + 1}} = \frac{1}{2}$$

يمكن وضع $t = x^2$ للتبسيط

$$\lim_{x \rightarrow -\infty} x\sqrt{x^2 + 1} + x^2 = \lim_{t \rightarrow +\infty} -t\sqrt{t^2 + 1} + t^2 = \lim_{t \rightarrow +\infty} t^2 - t\sqrt{t^2 + 1} = \lim_{t \rightarrow +\infty} \frac{t^4 - t^4 - t^2}{t^2 + t\sqrt{t^2 + 1}}$$

$$\lim_{x \rightarrow -\infty} x\sqrt{x^2 + 1} + x^2 = \lim_{t \rightarrow +\infty} \frac{-t^2}{t^2 + t^2 \sqrt{1 + \frac{1}{t^2}}} = \lim_{t \rightarrow +\infty} \frac{-1}{1 + \sqrt{1 + \frac{1}{t^2}}} = \frac{-1}{2} \quad (\text{نضع } t = -x)$$

$$\begin{aligned}
\lim_{x \mapsto \frac{f}{4}} \left(\frac{\sqrt{1-\cos(x)} - \sqrt{1-\sin(x)}}{1-\tan(x)} \right) &= \lim_{x \mapsto \frac{f}{4}} \frac{\sin(x) - \cos(x)}{(1-\tan(x))(\sqrt{1-\cos(x)} + \sqrt{1-\sin(x)})} \\
&= \lim_{x \mapsto \frac{f}{4}} \frac{\sqrt{2} \left(\frac{1}{\sqrt{2}} \sin(x) - \frac{1}{\sqrt{2}} \cos(x) \right)}{\left(\tan\left(\frac{f}{4} - x\right) (1 + \tan(x)) \right) (\sqrt{1-\cos(x)} + \sqrt{1-\sin(x)})} \\
&= \lim_{x \mapsto \frac{f}{4}} \frac{\sqrt{2} \left(\cos\left(\frac{f}{4}\right) \sin(x) - \sin\left(\frac{f}{4}\right) \cos(x) \right)}{\left(\tan\left(\frac{f}{4} - x\right) (1 + \tan(x)) \right) (\sqrt{1-\cos(x)} + \sqrt{1-\sin(x)})} \\
&= \lim_{x \mapsto \frac{f}{4}} \frac{\sqrt{2} \sin\left(x - \frac{f}{4}\right)}{\left(\tan\left(\frac{f}{4} - x\right) (1 + \tan(x)) \right) (\sqrt{1-\cos(x)} + \sqrt{1-\sin(x)})} \\
&= \lim_{x \mapsto \frac{f}{4}} \frac{-\sqrt{2}}{(1 + \tan(x)) (\sqrt{1-\cos(x)} + \sqrt{1-\sin(x)})} \left(\frac{\sin\left(x - \frac{f}{4}\right)}{x - \frac{f}{4}} \right) \left(\frac{x - \frac{f}{4}}{\tan\left(x - \frac{f}{4}\right)} \right) \\
&= \frac{-\sqrt{2}}{2 \times \left(\sqrt{1 - \frac{\sqrt{2}}{2}} + \sqrt{1 - \frac{\sqrt{2}}{2}} \right)} \times 1 \times 1 = \frac{-\sqrt{2}}{4 \sqrt{\frac{2-\sqrt{2}}{2}}} = \frac{-2}{4\sqrt{2-\sqrt{2}}} \\
\lim_{x \mapsto \frac{f}{4}} \left(\frac{\sqrt{1-\cos(x)} - \sqrt{1-\sin(x)}}{1-\tan(x)} \right) &= \frac{-1}{2} \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}} = \frac{-\sqrt{4+2\sqrt{2}}}{4}
\end{aligned}$$

تمرین 4 :

$$\lim_{x \mapsto a} \frac{x\sqrt{x} - a\sqrt{a}}{\sqrt{x} - \sqrt{a}} = \lim_{x \mapsto a} \frac{(\sqrt{x})^3 - (\sqrt{a})^3}{\sqrt{x} - \sqrt{a}} = \lim_{x \mapsto a} \frac{(\sqrt{x} - \sqrt{a}) (\sqrt{x}^2 + \sqrt{x}\sqrt{a} + \sqrt{a}^2)}{\sqrt{x} - \sqrt{a}} = \lim_{x \mapsto a} x + \sqrt{a}x + a = 3a$$

$$\begin{aligned}
\lim_{x \mapsto 1} \frac{x^{2015} - 1 - 2015(x-1)}{(x-1)^2} &= \lim_{x \mapsto 1} \frac{(x-1)(x^{2014} + x^{2013} + \dots + x + 1) - 2015(x-1)}{(x-1)^2} \\
&= \lim_{x \mapsto 1} \frac{x^{2014} + x^{2013} + \dots + x + 1 - 2015}{x-1} \\
&= \lim_{x \mapsto 1} \frac{x^{2014} + x^{2013} + \dots + x - 2014}{x-1} = \lim_{x \mapsto 1} \frac{x^{2014} - 1}{x-1} + \frac{x^{2013} - 1}{x-1} + \dots + \frac{x-1}{x-1} \\
&= \lim_{x \mapsto 1} (x^{2013} + x^{2012} + \dots + x + 1) + (x^{2012} + x^{2011} + \dots + x + 1) + \dots + 1
\end{aligned}$$

$$\lim_{x \mapsto 1} \frac{x^{2015} - 1 - 2015(x-1)}{(x-1)^2} = 2014 + 2013 + \dots + 1 = \frac{2014 \times 2015}{2} = 2029105$$