

تمرين 1: احسب النهايات التالية:

$$\lim_{x \rightarrow -\infty} (\sqrt{9x^2 + x + 1} + 3x)$$

$$\lim_{x \rightarrow +\infty} (\sqrt{9x^2 + x + 1} - 2x)$$

$$\lim_{x \rightarrow \infty} (\sqrt{1 + 2x^3} - \sqrt{x^3 + x + 1})$$

$$\lim_{x \rightarrow +\infty} (x^2 - \sqrt{x + 2})$$

$$\lim_{x \rightarrow +\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right)$$

$$m \in \mathbb{R} \text{ حيث } \lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx)$$

$$\lim_{x \rightarrow -\infty} \left(\frac{x + 1 - \sqrt{1 - x}}{x^2 - \sqrt{x^2 + 2}} \right)$$

$$\lim_{x \rightarrow -\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right)$$

تمرين 2: احسب النهايات التالية:

$$\lim_{x \rightarrow 2^+} \left(\frac{\sqrt{x^2 - 4}}{x - 2} \right)$$

$$\lim_{x \rightarrow -1} \left(\frac{\sqrt{1 - 3x} - 2}{x + 1} \right)$$

$$\lim_{x \rightarrow 1} \left(\frac{x^2 + x - 2}{x^3 + 4x^2 - 8x + 3} \right)$$

$$\lim_{x \rightarrow -2} \left(\frac{\sqrt{x^2 - 4}}{x + 2} \right)$$

$$\lim_{x \rightarrow 0^+} \left(\frac{x + \sqrt{x}}{\sqrt{x^2 + x} - x} \right)$$

$$\lim_{x \rightarrow -1} \left(\frac{\sqrt{2 + x} + \sqrt{3 - x} - 3}{x + 1} \right)$$

تمرين 3:

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{3} \cos x - \sin^2 x - \sqrt{3}} \right)$$

$$\lim_{x \rightarrow 0} \left(\frac{\sqrt{3} \cos x - \sin^2 x - \sqrt{3}}{x^2} \right)$$

$$\lim_{x \rightarrow \frac{f}{6}} \left(\frac{\cos x - \sqrt{3} \sin x}{6x - f} \right)$$

$$\lim_{x \rightarrow \frac{f}{4}} \left(\frac{\tan x - 1}{2 \cos x - \sqrt{2}} \right)$$

تمرين 4: نعتبر الدالة المعرفة كما يلي : $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ احسب $\lim_{x \rightarrow 0} f(x)$ ثم $\lim_{x \rightarrow +\infty} f(x)$

تمرين 1 :

$$\lim_{x \rightarrow +\infty} (\sqrt{9x^2 + x + 1} - 2x) = \lim_{x \rightarrow +\infty} \left(\sqrt{x^2 \left(9 + \frac{1}{x} + \frac{1}{x^2} \right)} - 2x \right) = \lim_{x \rightarrow +\infty} \left(x \sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 2x \right)$$

$$\lim_{x \rightarrow +\infty} (\sqrt{9x^2 + x + 1} - 2x) = \lim_{x \rightarrow +\infty} x \left(\sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 2 \right) = +\infty$$

(لأن : $\lim_{x \rightarrow +\infty} x = +\infty$ و $\lim_{x \rightarrow +\infty} \sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 2 = 3 - 2 = 1$)

$$\lim_{x \rightarrow -\infty} (\sqrt{9x^2 + x + 1} + 3x) = \lim_{x \rightarrow -\infty} \frac{9x^2 + x + 1 - 9x^2}{\sqrt{9x^2 + x + 1} - 3x} = \lim_{x \rightarrow -\infty} \frac{x + 1}{\sqrt{x^2 \left(9 + \frac{1}{x} + \frac{1}{x^2} \right)} - 3x}$$

$$\lim_{x \rightarrow -\infty} (\sqrt{9x^2 + x + 1} + 3x) = \lim_{x \rightarrow -\infty} \frac{x \left(1 + \frac{1}{x} \right)}{-x \sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 3x} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{-\sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 3} = \frac{1}{-3 - 3} = \frac{-1}{6}$$

انتبه أنه إذا كان : $x < 0$ فإن : $\sqrt{x^2} = |-x| = -x$

الطريقة السابقة لن تجدي في هذه النهاية لأننا سنحصل على شكل غير محدد

المثالان متشابهان لكن الطريقة تختلف و السبب المعاملات الموجودة في كل نهاية، لذلك نأمل أن تدرك متى نستعمل المرافق ومتى نعمل بأكبر أس.

$$\lim_{x \rightarrow +\infty} (x^2 - \sqrt{x+2}) = \lim_{x \rightarrow +\infty} \left(x^2 - \sqrt{x^2 \left(\frac{1}{x} + \frac{2}{x} \right)} \right) = \lim_{x \rightarrow +\infty} \left(x^2 - x \sqrt{\frac{1}{x} + \frac{2}{x}} \right) = \lim_{x \rightarrow +\infty} x \left(x - \sqrt{\frac{1}{x} + \frac{2}{x}} \right) = +\infty$$

(لأن : $\lim_{x \rightarrow +\infty} x = +\infty$ و $\lim_{x \rightarrow +\infty} x - \sqrt{\frac{1}{x} + \frac{2}{x}} = +\infty$)

$$\lim_{x \rightarrow +\infty} \sqrt{1+2x^3} - \sqrt{x^3+x+1} = \lim_{x \rightarrow +\infty} \sqrt{x^3} \sqrt{\frac{1}{x^3} + 2} - \sqrt{x^3} \sqrt{1 + \frac{1}{x^2} + \frac{1}{x^3}}$$

$$\lim_{x \rightarrow +\infty} \sqrt{1+2x^3} - \sqrt{x^3+x+1} = \lim_{x \rightarrow +\infty} \sqrt{x^3} \left(\sqrt{\frac{1}{x^3} + 2} - \sqrt{1 + \frac{1}{x^2} + \frac{1}{x^3}} \right) = +\infty$$

(لأن : $\lim_{x \rightarrow +\infty} \sqrt{x^3} = +\infty$ و $\lim_{x \rightarrow +\infty} \sqrt{\frac{1}{x^3} + 2} - \sqrt{1 + \frac{1}{x^2} + \frac{1}{x^3}} = \sqrt{2} - 1 > 0$)

■ إذا كان $m > 0$ فإن : $\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = +\infty$ ($\lim_{x \rightarrow -\infty} -mx = +\infty$)

■ إذا كان $m = 0$ فإن : $\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7}) = +\infty$

■ إذا كان $-2 < m < 0$ فإن :

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} \left(\sqrt{x^2 \left(4 + \frac{3}{x} + \frac{7}{x^2} \right)} - mx \right) = \lim_{x \rightarrow -\infty} \left(-x \sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} - mx \right)$$

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} -x \left(\sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} + m \right) = +\infty \quad (+\infty \times (2+m) > 0)$$

■ إذا كان $m = -2$ فإن :

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} \sqrt{4x^2 + 3x + 7} + 2x = \lim_{x \rightarrow -\infty} \frac{3x + 7}{\sqrt{4x^2 + 3x + 7} - 2x}$$

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} \frac{x \left(3 + \frac{7}{x}\right)}{-x \sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} - 2x} = \lim_{x \rightarrow -\infty} \frac{3 + \frac{7}{x}}{-\sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} - 2} = \frac{-3}{4}$$

إذا كان $m < -2$ فإن :

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} -x \left(\sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} + m \right) = -\infty \quad (+\infty \times (2+m) < 0)$$

سؤال يوضح بالتفاصيل متى نستعمل طريقة النهاية الأولى و متى نستعمل المرافق

$$\lim_{x \rightarrow +\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right) = \lim_{x \rightarrow +\infty} \frac{(x^2 - (x^2 + x + 1))(x^2 + \sqrt{x^4 - 1})}{(x^4 - (x^4 - 1))(x + \sqrt{x^2 + x + 1})} = \lim_{x \rightarrow +\infty} \frac{-(x+1) \left(x^2 + x^2 \sqrt{1 - \frac{1}{x^4}} \right)}{\left(x + x \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} \right)}$$

$$\lim_{x \rightarrow +\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right) = \lim_{x \rightarrow +\infty} \frac{-x(x+1) \left(1 + \sqrt{1 - \frac{1}{x^4}} \right)}{1 + \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}} = -\infty$$

$$\lim_{x \rightarrow -\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right) = \lim_{x \rightarrow -\infty} \frac{(x - \sqrt{x^2 + x + 1})(x^2 + \sqrt{x^4 - 1})}{(x^4 - (x^4 - 1))} = \lim_{x \rightarrow -\infty} (x - \sqrt{x^2 + x + 1}) \left(x^2 + x^2 \sqrt{1 - \frac{1}{x^4}} \right)$$

$$\lim_{x \rightarrow -\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right) = \lim_{x \rightarrow -\infty} x^2 (x - \sqrt{x^2 + x + 1}) \left(1 + \sqrt{1 - \frac{1}{x^4}} \right) = -\infty$$

(لأن : $\lim_{x \rightarrow -\infty} 1 + \sqrt{1 - \frac{1}{x^4}} = 1$ و $\lim_{x \rightarrow -\infty} x^2 = +\infty$ و $\lim_{x \rightarrow -\infty} x - \sqrt{x^2 + x + 1} = -\infty$ ($-\infty + -\infty$))

$$\lim_{x \rightarrow -\infty} \left(\frac{x+1 - \sqrt{1-x}}{x^2 - \sqrt{x^2+2}} \right) = \lim_{x \rightarrow -\infty} \frac{x+1 - \sqrt{x^2 \left(\frac{1}{x^2} - \frac{1}{x} \right)}}{x^2 - \sqrt{x^2 \left(1 + \frac{2}{x^2} \right)}} = \lim_{x \rightarrow -\infty} \frac{x+1 + x \sqrt{\frac{1}{x^2} - \frac{1}{x}}}{x^2 + x \sqrt{1 + \frac{2}{x^2}}}$$

$$= \lim_{x \rightarrow -\infty} \frac{x \left(1 + \frac{1}{x} + \sqrt{\frac{1}{x^2} - \frac{1}{x}} \right)}{x \left(x + \sqrt{1 + \frac{2}{x^2}} \right)} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x} + \sqrt{\frac{1}{x^2} - \frac{1}{x}}}{x + \sqrt{1 + \frac{2}{x^2}}} = 0$$

(لأن : $\lim_{x \rightarrow +\infty} 1 + \frac{1}{x} + \sqrt{\frac{1}{x^2} - \frac{1}{x}} = 1$ و $\lim_{x \rightarrow +\infty} x + \sqrt{1 + \frac{2}{x^2}} = -\infty$)

تمرين 2 :

$$\lim_{x \rightarrow -1} \left(\frac{\sqrt{1-3x-2}}{x+1} \right) = \lim_{x \rightarrow -1} \frac{1-3x-4}{(x+1)(\sqrt{1-3x+2})} = \lim_{x \rightarrow -1} \frac{-3}{\sqrt{1-3x+2}} = \frac{-3}{4}$$

$$\lim_{x \rightarrow 2^+} \frac{\sqrt{x^2-4}}{x-2} = \lim_{x \rightarrow 2^+} \sqrt{\frac{(x-2)(x+2)}{(x-2)^2}} = \lim_{x \rightarrow 2^+} \sqrt{\frac{x+2}{x-2}} = +\infty$$

$$\lim_{x \rightarrow -2^-} \frac{\sqrt{x^2 - 4}}{x + 2} = \lim_{x \rightarrow -2^-} \frac{-\sqrt{x^2 - 4}}{-(x + 2)} = \lim_{x \rightarrow -2^+} \frac{-\sqrt{(x-2)(x+2)}}{(x+2)^2} = \lim_{x \rightarrow -2^+} \frac{-\sqrt{x-2}}{\sqrt{x+2}} = -\infty$$

$$\lim_{x \rightarrow 1} \left(\frac{x^2 + x - 2}{x^3 + 4x^2 - 8x + 3} \right) = \lim_{x \rightarrow 1} \frac{(x-1)(x+2)}{(x-1)(x^2 + 5x - 3)} = 1$$

استعملنا القسمة الإقليدية على $(x-1)$ للتعميل في البسط و المقام

$$\begin{aligned} \lim_{x \rightarrow -1} \left(\frac{\sqrt{2+x} + \sqrt{3-x} - 3}{x+1} \right) &= \lim_{x \rightarrow -1} \left(\frac{\sqrt{2+x} - 1}{x+1} + \frac{\sqrt{3-x} - 2}{x+1} \right) \\ &= \lim_{x \rightarrow -1} \left(\frac{x+1}{(x+1)(\sqrt{2+x}+1)} + \frac{-1-x}{(x+1)(\sqrt{3-x}+2)} \right) \\ &= \lim_{x \rightarrow -1} \left(\frac{1}{\sqrt{2+x}+1} + \frac{-1}{\sqrt{3-x}+2} \right) \end{aligned}$$

$$\lim_{x \rightarrow -1} \left(\frac{\sqrt{2+x} + \sqrt{3-x} - 3}{x+1} \right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\lim_{x \rightarrow 0^+} \left(\frac{x + \sqrt{x}}{\sqrt{x^2 + x} - x} \right) = \lim_{x \rightarrow 0^+} \left(\frac{\sqrt{x^2 + x}}{\sqrt{x(x+1)} - x} \right) = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}(\sqrt{x+1})}{\sqrt{x}\sqrt{x+1} - \sqrt{x^2}} = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}(\sqrt{x+1})}{\sqrt{x}(\sqrt{x+1} - \sqrt{x})}$$

$$\lim_{x \rightarrow 0^+} \left(\frac{x + \sqrt{x}}{\sqrt{x^2 + x} - x} \right) = \lim_{x \rightarrow 0^+} \frac{\sqrt{x} + 1}{\sqrt{x+1} - \sqrt{x}} = 1$$

تمرين 3 :

$$\lim_{x \rightarrow 0} \left(\frac{\sqrt{3} \cos x - \sin^2 x - \sqrt{3}}{x^2} \right) = \lim_{x \rightarrow 0} \left(-\sqrt{3} \left(\frac{1 - \cos(x)}{x^2} \right) - \left(\frac{\sin(x)}{x} \right)^2 \right) = \frac{-\sqrt{3}}{2} - 1$$

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{3} \cos x - \sin^2 x - \sqrt{3}} \right) = \lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{3}(\cos x - 1) - \sin^2 x} \right) = \lim_{x \rightarrow 0} \left(\frac{-1}{\sqrt{3}(1 - \cos(x)) + \sin^2 x} \right)$$

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{3} \cos x - \sin^2 x - \sqrt{3}} \right) = \lim_{x \rightarrow 0} \left(\frac{-\frac{1}{x^2}}{\sqrt{3} \left(\frac{1 - \cos(x)}{x^2} \right) + \left(\frac{\sin x}{x} \right)^2} \right) = -\infty$$

للتذكير: $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 1$ و $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\lim_{x \rightarrow \frac{f}{4}} \left(\frac{\tan x - 1}{2 \cos x - \sqrt{2}} \right) = \lim_{x \rightarrow \frac{f}{4}} \left(\frac{\tan x - \tan\left(\frac{f}{4}\right)}{2 \cos x - \sqrt{2}} \right) = \lim_{x \rightarrow \frac{f}{4}} \left(\frac{\left(1 + \tan(x)\tan\left(\frac{f}{4}\right)\right)\tan\left(x - \frac{f}{4}\right)}{2 \cos x - \sqrt{2}} \right)$$

لدينا :

الآن نضع $t = x - \frac{f}{4}$ فنجد :

$$\lim_{x \rightarrow \frac{f}{4}} \left(\frac{\tan x - 1}{2 \cos x - \sqrt{2}} \right) = \lim_{t \rightarrow 0} \left(\frac{\left(1 + \tan\left(t + \frac{f}{4}\right)\right) \tan(t)}{2 \cos\left(t + \frac{f}{4}\right) - \sqrt{2}} \right) = \lim_{t \rightarrow 0} \left(\frac{\left(1 + \tan\left(t + \frac{f}{4}\right)\right) \tan(t)}{2 \left(\cos(t) \frac{\sqrt{2}}{2} - \sin(t) \frac{\sqrt{2}}{2} \right) - \sqrt{2}} \right)$$

$$= \lim_{t \rightarrow 0} \left(\frac{\left(1 + \tan\left(t + \frac{f}{4}\right)\right) \tan(t)}{-\sqrt{2}(1 - \cos(t)) - \sqrt{2} \sin(t)} \right) = \lim_{t \rightarrow 0} \left(\frac{\left(1 + \tan\left(t + \frac{f}{4}\right)\right) \frac{\tan(t)}{t}}{-\sqrt{2} t \frac{(1 - \cos(t))}{t^2} - \sqrt{2} \frac{\sin(t)}{t}} \right)$$

$$\lim_{x \rightarrow \frac{f}{4}} \left(\frac{\tan x - 1}{2 \cos x - \sqrt{2}} \right) = \frac{(1+1) \times 1}{-\sqrt{2} \times 0 \times \frac{1}{2} - \sqrt{2} \times 1} = -\sqrt{2}$$

🍌 لاحظ أن استعمال الخاصية $\tan(a-b) = \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}$

بالشكل $\tan(a) - \tan(b) = (1 + \tan(a)\tan(b))\tan(a-b)$ أفضل من استعمال تغيير المتغير من البداية.

$$\lim_{x \rightarrow \frac{f}{6}} \left(\frac{\cos x - \sqrt{3} \sin x}{6x - f} \right) = \lim_{x \rightarrow \frac{f}{6}} \left(\frac{2 \left(\frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x \right)}{6 \left(x - \frac{f}{6} \right)} \right) = \lim_{x \rightarrow \frac{f}{6}} \left(\frac{1}{3} \frac{\left(\sin\left(\frac{f}{6}\right) \cos x - \cos\left(\frac{f}{6}\right) \sin x \right)}{\left(x - \frac{f}{6} \right)} \right)$$

$$\lim_{x \rightarrow \frac{f}{6}} \left(\frac{\cos x - \sqrt{3} \sin x}{6x - f} \right) = \lim_{x \rightarrow \frac{f}{6}} \left(\frac{1}{3} \frac{\sin\left(\frac{f}{6} - x\right)}{\left(x - \frac{f}{6} \right)} \right) = \lim_{x \rightarrow \frac{f}{6}} \left(\frac{-1}{3} \frac{\sin\left(x - \frac{f}{6}\right)}{\left(x - \frac{f}{6} \right)} \right) = \lim_{t \rightarrow 0} \left(\frac{-1}{3} \frac{\sin(t)}{t} \right) = \frac{-1}{3}$$

🍌 (قمنا بتغيير المتغير x وذلك بوضع $t = x - \frac{f}{6}$)، كما يمكن إجراء تغيير المتغير منذ البداية.

تمرين 4: $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$

لدينا: $\forall x \in \mathbb{R}^* \quad -x^2 \leq f(x) \leq x^2$ منه: $\forall x \in \mathbb{R}^* \quad -1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$

بما أن: $\lim_{x \rightarrow 0} x^2 = 0$ و $\lim_{x \rightarrow 0} -x^2 = 0$ فإن: $\lim_{x \rightarrow 0} f(x) = 0$

نضع: $t = \frac{1}{x^2}$ إذن: $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} x^2 \sin\left(\frac{1}{x^2}\right) = \lim_{t \rightarrow 0} \frac{1}{t} \sin(t) = 1$